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Dynamical and kinematical supersymmetries of the quantum harmonic oscillator and the motion in a constant magnetic field†

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Abstract. Dynamical and kinematical symmetries and supersymmetries of the n-dimensional harmonic oscillator are discussed in connection with two different supersymmetrisation procedures: the so-called standard and spin-orbit coupling procedures. The largest invariance structures appear within the standard procedure dealing with the same numbers of bosonic and fermionic degrees of freedom. We also get meaningful substructures within the spin-orbit coupling procedure dealing with fermionic degrees of freedom which are half of the bosonic ones. Finally the $n=2$ case is connected with the study of the motion in a constant magnetic field. Each procedure leads to a nice correspondence between the two systems under consideration.

1. Introduction

We recently discussed (Beckers *et al* 1987) the supersymmetric version of the one-dimensional harmonic oscillator by taking into account its conformal properties. This paper will hereafter be referred to as I and for brevity we will refer to some of its equations. We mainly plan to extend these considerations to *dynamical* as well as *kinematical* symmetries and supersymmetries.

Let us recall that a *dynamical* group (Wybourne 1974) of a physical system is a symmetry group that can yield the energy spectrum and the degeneracies of the levels and that contains a set of operators determining the transition probabilities between states. The study of such dynamical groups has already been considered in connection with elementary particle spectroscopy (Barut 1964, 1965, 1972) and with fundamental applications in quantum mechanics (Wybourne 1974), the harmonic oscillator and the hydrogen atom, for example.

A *kinematical* group or, more precisely, a maximal kinematical invariance group (Niederer 1972) is the largest group of spacetime transformations which leave invariant the wave equation describing a physical system. In the relativistic context the Poincaré group (Wigner 1939) and the conformal group of spacetime (Kastrup 1962, Mack and Salam 1969) are the maximal kinematical invariance groups of free non-zero and zero mass particles, respectively. In the non-relativistic context Niederer (1972) and Hagen

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(1972) have determined the maximal kinematical invariance group of a three-dimensional free particle as a twelve-parameter Lie group containing the usual Galilean transformations as well as the dilatations and expansions and characterising the so-called 'non-relativistic conformal quantum mechanics'. This latter domain has been extensively studied in connection with different specific Schrödinger equations with interaction (Niederer 1973, 1974, Boyer 1974) where we evidently notice a particular interest for the harmonic oscillator.

Dynamical as well as kinematical groups of the harmonic oscillator have some specific and some common properties. If the dynamical ones show remarkable features in connection with the symplectic groups (Wybourne 1974, Jacobson 1962, Gilmore 1974), the dynamical *and* kinematical groups have in common the Heisenberg group (Miller 1968, Talman 1968). Moreover, these Heisenberg groups play a primordial role since their associated algebras are generated by the bosonic creation and annihilation operators.

All the above notions and structures can be analysed and extended to the recent context of supersymmetric quantum mechanics (Witten 1981). In fact, different contributions have already been published (de Crombrugghe and Rittenberg 1983, Fubini and Rabinovici 1984, Durand 1985, Balantekin 1985, Kostelecky *et al* 1985, Gamboa and Zanelli 1985, Beckers and Hussin 1986, D'Hoker *et al* 1987) if we limit ourselves, besides I, mainly to those dealing with the harmonic oscillator case and with the non-relativistic conformal (super)symmetries. For the dynamical point of view we especially point out the de Crombrugghe–Rittenberg paper in connection with the so-called standard procedure of supersymmetrisation (Witten 1981) applied to the harmonic oscillator.

Our purposes here are twofold. Firstly, we want to point out the properties of the non-supersymmetric n-dimensional harmonic oscillator in what concerns its dynamical and kinematical symmetries. This discussion will then be extended to its supersymmetries by studying and comparing two different procedures of supersymmetrisation: the standard one (Witten 1981) and the spin–orbit coupling one introduced and discussed in I. Secondly, we want to take advantage of the particular case of two spatial dimensions and to connect it with the study of symmetries and supersymmetries for the motion in a constant magnetic field, another fundamental application in (supersymmetric) quantum mechanics.

This second purpose needs a few more comments in order to point out its interest. Since the work of Johnson and Lippmann (1949) the study of the motion in a constant magnetic field has been related to typical properties of the harmonic oscillator. In two-dimensional spaces, the Hamiltonians of the harmonic oscillator and of the above magnetic case are the generators of two non-conjugate one-dimensional subalgebras of the inhomogeneous symplectic invariance algebra (Burdet and Perrin 1975). Moreover we have just constructed a time-dependent rotation (Dehin and Hussin 1987) putting these two applications in a one-to-one correspondence as Niederer (1973) gave an *ad hoc* change of variables connecting the harmonic oscillator to the free case. Finally we also recall (Boyer 1974) that the $n = 2$ context has a direct bearing on the study of conformal invariance in *relativistic* mechanics when viewed from the infinite momentum frame (Domokos 1972, Burdet *et al* 1973). Their supersymmetric versions then appear to be very interesting with, among other purposes, the one corresponding to the construction of the relativistic supersymmetric *quantum* mechanics.

The contents of this paper are arranged as follows. In § 2, the n-dimensional quantum harmonic oscillator is characterised by its dynamical (Wybourne 1974) and kinematical (Niederer 1973) symmetries (§ 2.1) as well as supersymmetries (§§ 2.2 and 2.3), the last two subsections being devoted to the discussion of the standard and spin-orbit coupling procedures of supersymmetrisation. Here we will effectively put the accent on the corresponding algebras and superalgebras as well as on their inclusion properties. Restricted to the $(n = 2)$-dimensional case, § 3 concerns the study of the motion in a constant magnetic field chosen along the z axis and its connection with the harmonic oscillator case. Once again its symmetries (§ 3.1) as well as its supersymmetries (§§ 3.2, 3.3 and 3.4) are considered, the latter being discussed (§§ 3.2 and 3.3) and compared (§ 3.4) in connection with the above two supersymmetrisation procedures. Finally the change of variables (Dehin and Hussin 1987) enlightening the connection between the two-dimensional harmonic oscillator and the motion in a constant magnetic field (already applied to symmetries) is considered in § 4 in the supersymmetric context dealing with both bosonic and fermionic variables.

According to those used in I, our notations and conventions are the current ones but let us mention that *all* the (super)algebras specified in this paper are defined over the real field $\mathbb{R}$. For brevity we have then suppressed this symbol. Moreover, with the conventional notations for the (super)symplectic structures sp$(2n)$, sp(4), osp$(2n/2n)$ and osp(4/4), we point out that the spin-orbit coupling procedure leads to a superalgebra osp(2/4) where the fermionic degrees of freedom are half the bosonic ones. Accordingly we decide to refer to the corresponding Heisenberg (super)algebras by denoting them as follows: h$(2n)$, h(4), sh$(2n/2n)$ and sh(4/4) but by sh(2/4) for the Heisenberg superalgebra of the spin-orbit coupling procedure. The dimensions of all these (super)algebras will be explicitly given in the text.

2. Symmetries and supersymmetries of the quantum harmonic oscillator

Let us study here the harmonic oscillator and its (super)symmetries when the n-dimensional case is considered. The symmetries (§ 2.1) refer to *dynamical* ones following Wybourne (1974) as well as to *kinematical* ones following essentially Niederer's approach (Niederer 1973). The supersymmetries arising from the standard procedure (Witten 1981) (§ 2.2) are essentially deduced from the de Crombrugghe-Rittenberg contribution (de Crombrugghe and Rittenberg 1983) looking for the largest dynamical superalgebra. Then we present the so-called spin-orbit coupling procedure (Balantekin 1985, Kostelecky *et al* 1985, Gamboa and Zanelli 1985, Beckers *et al* 1987) (§ 2.3) and the corresponding results.

2.1. The n-dimensional harmonic oscillator

The non-supersymmetric n-dimensional harmonic oscillator is characterised when the mass m is taken as unity by the Hamiltonian

$$H_{\mathrm{O}} = \tfrac{1}{2}(\boldsymbol{p}^2 + \omega^2 \boldsymbol{x}^2) = \tfrac{1}{2}\omega\{\boldsymbol{a}_+, \boldsymbol{a}_-\} \tag{2.1}$$

where the subscript O refers to the oscillator case and where n-dimensional vectors in general as well as current creation $(\boldsymbol{a}_+)$ and annihilation $(\boldsymbol{a}_-)$ operators are denoted as usual by $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$. These operators are defined by

$$a_{\pm,k} = \frac{1}{\sqrt{2\omega}}(\omega x_k \mp \mathrm{i} p_k) \qquad k = 1, \ldots, n \tag{2.2}$$

and satisfy the commutation relations

$$[a_{-,k}, a_{+,l}] = \delta_{kl}. \tag{2.3}$$

Wybourne (1974) has determined the maximal *dynamical* invariance (MDI) algebra as the one associated with the degeneracy group of the harmonic oscillator. The MDI algebra is found to be $\mathrm{sp}(2n) \oplus \mathrm{h}(2n)$ with $(n+1)(2n+1)$ dimensions, the semi-direct sum of the $n(2n+1)$-dimensional symplectic algebra $\mathrm{sp}(2n)$ and the $(2n+1)$-dimensional Heisenberg algebra $\mathrm{h}(2n)$. Its generators correspond to the constants of motion for the Hamiltonian (2.1) and take the form:

$$P_{\pm,k}(t) = \pm\mathrm{i}\exp(\mp\mathrm{i}\omega t)(\omega x_k \mp \mathrm{i}p_k) = \pm\mathrm{i}(2\omega)^{1/2}\exp(\mp\mathrm{i}\omega t)a_{\pm,k} \tag{2.4}$$

$$\begin{aligned} T_{kl}(t) &= T_{kl}(0) = \tfrac{1}{2}\omega\{a_{-,k}, a_{+,l}\} = \tfrac{1}{4}\{P_{-,k}(t), P_{+,l}(t)\} \\ C_{\pm,kl}(t) &= \pm\tfrac{1}{2}\mathrm{i}\omega\exp(\mp 2\mathrm{i}\omega t)\{a_{\pm,k}, a_{\pm,l}\} = \mp\tfrac{1}{4}\mathrm{i}\{P_{\pm,k}(t), P_{\pm,l}(t)\}. \end{aligned} \tag{2.5}$$

Together with the identity operator, the $P_{\pm,k} \equiv (2.4)$ generate the algebra $\mathrm{h}(2n)$ while the T and $C_\pm$ given by (2.5) generate the algebra $\mathrm{sp}(2n)$. Let us insist here on the fundamental role played by the Heisenberg algebra: the generators (2.5) appear as combinations of second-order products of the generators (2.4). Their commutation relations are easily obtained by taking into account the relations (2.3).

After Niederer (1973) we know that the maximal *kinematical* invariance (MKI) algebra of the harmonic oscillator is determined by the generators associated with coordinate transformations, leaving invariant the corresponding Schrödinger equation in n dimensions. Such an algebra has the structure $[\mathrm{so}(2,1)\oplus\mathrm{so}(n)]\oplus\mathrm{h}(2n)$; it is of dimension $(3+\frac{1}{2}n(n-1)+2n+1)$ and is isomorphic to $\widetilde{\mathrm{schr}}(n)$, the central extension of the Schrödinger algebra, the MKI algebra (Niederer 1972) of the free Schrödinger equation. These considerations refer to the so-called 'conformal non-relativistic quantum mechanics' (Hagen 1972, Niederer 1972). The $\mathrm{so}(2,1)$ algebra corresponds to the (three) conformal symmetries including the Hamiltonian (2.1) and the generators

$$C_\pm(t) = \pm\mathrm{i}\omega\exp(\mp 2\mathrm{i}\omega t)\boldsymbol{a}_\pm^2 = \mp\tfrac{1}{2}\mathrm{i}\boldsymbol{P}_\pm^2(t). \tag{2.6}$$

The $\mathrm{so}(n)$ algebra is generated by the $\frac{1}{2}n(n-1)$ operators associated with spatial rotations; they are given by

$$\begin{aligned} L_{kl} &= x_k p_l - x_l p_k = \mathrm{i}(a_{-,k}a_{+,l} - a_{-,l}a_{+,k}) \\ &= (\mathrm{i}/2\omega)(P_{-,k}P_{+,l} - P_{-,l}P_{+,k}). \end{aligned} \tag{2.7}$$

Let us consequently point out the following inclusion:

$$[\mathrm{sp}(2n)\oplus\mathrm{h}(2n)] \supseteq [\mathrm{so}(2,1)\oplus\mathrm{so}(n)]\oplus\mathrm{h}(2n) \tag{2.8}$$

the equality only being ensured for the one-dimensional case. In fact, we have (with summation on repeated indices)

$$H_{\mathrm{O}} = T_{kk} \qquad C_\pm = C_{\pm,kk} \qquad L_{kl} = (\mathrm{i}/\omega)(T_{kl} - T_{lk}) \tag{2.9}$$

and we notice that, besides the kinematical generators, there are dynamical ones which cannot be associated with coordinate transformations.

2.2. *The n-dimensional supersymmetric harmonic oscillator and the standard procedure*

Within the supersymmetric (SS) version described by de Crombrugghe and Rittenberg

(1983), the harmonic oscillator Hamiltonian is

$$H_{\mathrm{O}}^{\mathrm{SS}}=\tfrac{1}{2}(\boldsymbol{p}^2+\omega^2\boldsymbol{x}^2)+\tfrac{1}{2}\omega[\boldsymbol{\xi}_+,\boldsymbol{\xi}_-]$$
$$=\tfrac{1}{2}\omega\{\boldsymbol{a}_+,\boldsymbol{a}_-\}+\tfrac{1}{2}\omega[\boldsymbol{\xi}_+,\boldsymbol{\xi}_-] \tag{2.10}$$

where the operators $\xi_{\pm,k}$ are Grassmann variables satisfying

$$\{\xi_{+,k},\xi_{-,l}\}=\delta_{kl} \qquad \{\xi_{\pm,k},\xi_{\pm,l}\}=0. \tag{2.11}$$

With the choice of the fermionic quantities

$$\varphi_k^1=\xi_{+,k}+\xi_{-,k} \qquad \varphi_k^2=\mathrm{i}(\xi_{-,k}-\xi_{+,k}) \tag{2.12}$$

we obtain a Clifford algebra

$$\{\varphi_k^a,\varphi_l^b\}=2\delta^{ab}\delta_{kl}. \tag{2.13}$$

We then deal with a description which admits the same number $(2n)$ of bosonic and fermionic degrees of freedom, the bosonic ones being the canonically conjugated (generalised) positions and momenta.

The supersymmetry subtended by these considerations is a $N=2$ supersymmetry with two type-Q supercharges (see equations (I, 6.6))

$$Q_\pm=\frac{1}{\sqrt{2}}(\boldsymbol{p}\mp\mathrm{i}\omega\boldsymbol{x})\cdot\boldsymbol{\xi}_\pm=\mp\mathrm{i}\omega^{1/2}\boldsymbol{a}_\mp\cdot\boldsymbol{\xi}_\pm \tag{2.14}$$

such that

$$\{Q_+,Q_-\}=\mathrm{H}_{\mathrm{O}}^{\mathrm{SS}} \tag{2.15}$$

and

$$[Q_\pm,H_{\mathrm{O}}^{\mathrm{SS}}]=0. \tag{2.16}$$

Here the supersymmetric version of the MDI algebra $\mathrm{sp}(2n)\oplus\mathrm{h}(2n)$ is the $(8n^2+4n+1)$-dimensional MDI superalgebra $\mathrm{osp}(2n/2n)\oplus\mathrm{sh}(2n/2n)$, the semi-direct sum of the $(8n^2)$-dimensional superalgebra $\mathrm{osp}(2n/2n)$ and the $(4n+1)$-dimensional Heisenberg superalgebra $\mathrm{sh}(2n/2n)$. We evidently have the inclusion

$$[\mathrm{osp}(2n/2n)\oplus\mathrm{sh}(2n/2n)]\supset[\mathrm{sp}(2n)\oplus\mathrm{h}(2n)]. \tag{2.17}$$

Let us insist on the fact that the Heisenberg superalgebra is generated by the identity operator, the $P_{\pm,k}$ given by (2.4) and their fermionic analogous (see equations (I, 6.11)):

$$T_{\pm,k}(t)=\exp(\mp\mathrm{i}\omega t)\xi_{\pm,k} \qquad k=1,2,\ldots,n. \tag{2.18}$$

With respect to the superalgebra $\mathrm{osp}(2n/2n)$ we point out that it is constructed from bosonic symmetries associated with $\mathrm{sp}(2n)$ given in (2.5), from fermionic symmetries described by the algebra $\mathrm{so}(2n)$ generated by $\{Y_{kl}(t),\ Z_{\pm,kl}(t),\ k,l=1,2,\ldots,n\}$ and, finally, from supersymmetries with type-Q and type-S (Fubini and Rabinovici 1984) supercharges $\{Q_{\pm,kl}(t),\ S_{\pm,kl}(t)\}$. Explicitly these operators are

$$\begin{aligned} Y_{kl}(t)&=Y_{kl}(0)=\tfrac{1}{2}\omega[\xi_{+,k},\xi_{-,l}]=\tfrac{1}{2}\omega[T_{+,k},T_{-,l}]\\ Z_{\pm,kl}(t)&=\pm\tfrac{1}{2}\mathrm{i}\omega\exp(\mp2\mathrm{i}\omega t)[\xi_{\pm,k},\xi_{\pm,l}]=\pm\tfrac{1}{2}\mathrm{i}\omega[T_{\pm,k},T_{\pm,l}]\end{aligned} \tag{2.19}$$

and

$$\begin{aligned} Q_{\pm,kl}(t)&=Q_{\pm,kl}(0)=\mp\mathrm{i}\omega^{1/2}a_{\mp,k}\xi_{\pm,l}=\frac{1}{\sqrt{2}}P_{\mp,k}T_{\pm,l}\\ S_{\pm,kl}(t)&=\pm\mathrm{i}\omega^{1/2}\exp(\mp2\mathrm{i}\omega t)a_{\pm,k}\xi_{\pm,l}=\frac{1}{\sqrt{2}}P_{\pm,k}T_{\pm,l}.\end{aligned} \tag{2.20}$$

Such characteristics essentially correspond to the de Crombrugghe–Rittenberg results which, however, did not contain our Heisenberg superalgebra sh$(2n/2n)$. Let us notice that this last superalgebra appears to be fundamental as already shown in I since the osp$(2n/2n)$ generators are written as second-order products of the sh$(2n/2n)$ operators (2.4) and (2.18).

Now, in connection with I, if we want to limit ourselves to *kinematical* symmetries *and* supersymmetries of the harmonic oscillator, i.e. if we want to determine the MKI superalgebra enhancing coordinate transformation laws, we immediately notice that the symmetries of $H_O \equiv (2.1)$ generating the algebra $[\mathrm{so}(2,1)\oplus\mathrm{so}(n)]\oplus\mathrm{h}(2n)$ are still conserved for the Hamiltonian $H_O^{SS} \equiv (2.10)$ but have to be completed by others in order to close the superalgebra. In fact, besides the generators $H_O \equiv (2.1)$, $P_{\pm,k} \equiv (2.4)$, $C_\pm \equiv (2.6)$ associated with kinematical *bosonic* symmetries as well as the supercharges (2.14), we have to add the generators associated with the *fermionic* symmetries $T_{\pm,k} \equiv$ (2.18) and Y given by

$$Y = \tfrac{1}{2}\omega[\boldsymbol{\xi}_+, \boldsymbol{\xi}_-] \tag{2.21}$$

as well as with the type-S supercharges (see equations (I, 6.12))

$$S_\pm = \frac{1}{\sqrt{2}} \exp(\mp 2\mathrm{i}\omega t)(\boldsymbol{p} \pm \mathrm{i}\omega \boldsymbol{x}) \cdot \boldsymbol{\xi}_\pm = \pm\mathrm{i}\omega^{1/2} \exp(\mp 2\mathrm{i}\omega t)\boldsymbol{a}_\pm \cdot \boldsymbol{\xi}_\pm . \tag{2.22}$$

Finally we have to replace the generators $L_{kl} \equiv (2.7)$ (corresponding only to *orbital* angular momentum considerations) by new ones corresponding to the *total* angular momentum, i.e.

$$J_{kl} = L_{kl} + \Sigma_{kl} \tag{2.23}$$

where

$$\Sigma_{kl} = -\mathrm{i}(\xi_{-,k}\xi_{+,l} - \xi_{-,l}\xi_{+,k}). \tag{2.24}$$

It is easy to recognise that the whole set of the above generators corresponds to the MKI superalgebra $[\mathrm{osp}(2/2)\oplus\mathrm{so}(n)]\oplus\mathrm{sh}(2n/2n)$ where, in particular, the eight-dimensional superalgebra osp(2/2) is generated by the three conformal operators $\{H_O, C_\pm\}(\sim\mathrm{so}(2,1))$, by the fermionic operator $Y(\in\mathrm{so}(2))$ and by the four supercharges $Q_\pm$ and $S_\pm$. The osp(2/2) commutation and anticommutation relations are summarised in the form

$$\begin{aligned}
&[H_O, C_\pm] = \pm 2\omega C_\pm \qquad [C_+, C_-] = -4\omega H_O \\
&[H_O, Q_\pm] = \mp\omega Q_\pm = [Q_\pm, Y] \qquad [H_O, S_\pm] = \pm\omega S_\pm = [Y, S_\pm] \\
&[C_\pm, Q_\pm] = 2\mathrm{i}\omega S_\pm \qquad [C_\pm, S_\mp] = 2\mathrm{i}\omega Q_\mp \\
&\{Q_+, Q_-\} = H_O + Y = H_O^{SS} \qquad \{S_+, S_-\} = H_O - Y \\
&\{Q_\pm, S_\mp\} = \mp\mathrm{i}C_\mp .
\end{aligned} \tag{2.25}$$

The further structure relations of the MKI superalgebra are

$$\begin{aligned}
&[H_O, P_{\pm,k}] = \pm\omega P_{\pm,k} \qquad [C_\pm, P_{\mp,k}] = 2\mathrm{i}\omega P_{\pm,k} \\
&[P_{-,k}, P_{+,l}] = 2\omega\delta_{kl} I \\
&[Y, T_{\pm,k}] = \pm\omega T_{\pm,k} \qquad \{T_{+,k}, T_{-,l}\} = \delta_{kl} I
\end{aligned}$$

$$\begin{aligned}
&[J_{kl}, J_{mn}] = \mathrm{i}(\delta_{km}J_{ln} + \delta_{ln}J_{km} - \delta_{kn}J_{lm} - \delta_{lm}J_{kn}) \\
&[J_{kl}, P_{\pm,m}] = \mathrm{i}(\delta_{km}P_{\pm,l} - \delta_{lm}P_{\pm,k}) \\
&[J_{kl}, T_{\pm,m}] = \mathrm{i}(\delta_{km}T_{\pm,l} - \delta_{lm}T_{\pm,k}) \\
&[Q_\pm, P_{\pm,k}] = \pm\sqrt{2}\omega T_{\pm,k} \qquad [S_\pm, P_{\mp,k}] = \mp\sqrt{2}\omega T_{\pm,k} \\
&\{Q_\pm, T_{\mp,k}\} = \frac{1}{\sqrt{2}}P_{\mp,k} \qquad \{S_\pm, T_{\mp,k}\} = \frac{1}{\sqrt{2}}P_{\pm,k}.
\end{aligned} \tag{2.26}$$

Such a kinematical superalgebra is evidently contained in the dynamical superalgebra previously discussed. We have

$$[\mathrm{osp}(2n/2n) \oplus \mathrm{sh}(2n/2n)] \supseteq [[\mathrm{osp}(2/2) \oplus \mathrm{so}(n)] \oplus \mathrm{sh}(2n/2n)] \tag{2.27}$$

the equality only being ensured for the one-dimensional context showing that dynamical and kinematical supersymmetries then coincide (see I). As a last remark let us mention that the fermionic generators (2.21) and (2.24), as well as the supercharges (2.14) and (2.22), can be simply expressed in terms of the dynamical operators (2.19) and (2.20). Indeed we have

$$\begin{aligned}
&Y = Y_{kk} \qquad \Sigma_{kl} = -(\mathrm{i}/\omega)(Y_{kl} - Y_{lk}) \\
&Q_\pm = Q_{\pm,kk} \qquad S_\pm = S_{\pm,kk}.
\end{aligned} \tag{2.28}$$

Due to the definitions (2.4) and (2.18) we also get

$$Y = \tfrac{1}{2}\omega[T_{+,k}, T_{-,k}] \qquad \Sigma_{kl} = -\mathrm{i}(T_{-,k}T_{+,l} - T_{-,l}T_{+,k}) \tag{2.29}$$

and

$$Q_\pm = \frac{1}{\sqrt{2}}\boldsymbol{P}_\mp \cdot \boldsymbol{T}_\pm \qquad S_\pm = \frac{1}{\sqrt{2}}\boldsymbol{P}_\pm \cdot \boldsymbol{T}_\mp. \tag{2.30}$$

2.3. *The n-dimensional supersymmetric harmonic oscillator and the spin–orbit coupling procedure*

Now let us comment on another supersymmetrisation procedure we have proposed (see I) by modifying the structure relations (2.11) or (2.13) on the variables $\xi_{\pm,k}$ or φ_k^a. Let us require

$$\begin{aligned}
&\{\xi_{+,k}, \xi_{-,l}\} = \delta_{kl} - \mathrm{i}\Xi_{kl} \qquad \Xi_{kl} = -\Xi_{lk} \qquad (\Xi)^+ = \Xi \\
&\{\xi_{\pm,k}, \xi_{\pm,l}\} = 0 \qquad k, l = 1, 2, \ldots, n
\end{aligned} \tag{2.31}$$

corresponding, on the variables φ_k^a, to the algebra

$$\{\varphi_k^a, \varphi_l^b\} = 2(\delta^{ab}\delta_{kl} + \varepsilon^{ab}\Xi_{kl}) \qquad \varepsilon^{12} = -\varepsilon^{21} = 1. \tag{2.32}$$

This relation essentially differs from the one obtained for the Clifford algebra (2.13) and leads to a smaller number of fermionic degrees of freedom in this procedure.

We can then develop a supersymmetric theory characterised by the same supercharges $Q_\pm$ given by (2.14) but leading to a new Hamiltonian $(H_{\mathrm{O}}^{\mathrm{SS}})'$. We effectively obtain with (2.31)

$$\{Q_+, Q_-\} = (H_{\mathrm{O}}^{\mathrm{SS}})'$$

where

$$(H_{\mathrm{O}}^{\mathrm{SS}})' = \tfrac{1}{2}(\boldsymbol{p}^2 + \omega^2 \boldsymbol{x}^2) + \tfrac{1}{2}\omega[\boldsymbol{\xi}_+, \boldsymbol{\xi}_-] - \tfrac{1}{2}\omega(x_k p_l - x_l p_k)\Xi_{kl}. \tag{2.33}$$

Such a Hamiltonian contains the usual bosonic part (see (2.1)) but its fermionic part is supplemented by a spin-orbit coupling term so that we recover the supersymmetric Hamiltonian recently discussed by Balantekin (1985) in the three-dimensional context and generalised to the n-dimensional one by Kostelecky *et al* (1985). In fact, we get from (2.10) and (2.33)

$$(H_{\mathrm{O}}^{\mathrm{SS}})' = H_{\mathrm{O}}^{\mathrm{SS}} - \tfrac{1}{2}\omega L_{kl}\Xi_{kl}. \tag{2.34}$$

The symmetries and supersymmetries of the Hamiltonian $(H_{\mathrm{O}}^{\mathrm{SS}})' \equiv (2.34)$ could be discussed in connection with those of $H_{\mathrm{O}}^{\mathrm{SS}} \equiv (2.10)$. They differ by the presence of the additional spin-orbit coupling term justifying the appellation of such a supersymmetrisation procedure.

Here let us notice that the superalgebra osp(2/2)$\oplus$so(3) obtained by Balantekin (1985) as the MKI superalgebra of the Hamiltonian (2.34) *in the three-dimensional case* has also been generalised by Kostelecky *et al* (1985) to the n-dimensional context. While we start from supercharges leading to the Hamiltonian (2.34), Kostelecky *et al* (1985) start from a *given* Hamiltonian with spin-orbit coupling term and show that it is supersymmetric. Indeed it admits supercharges $Q_\pm \equiv (2.14)$ but with a specific realisation of our variables $\xi_{\pm,k}$. In fact, it is easy to see that

$$\xi_{\pm,k} = \Gamma_k \otimes \sigma_\pm \tag{2.35}$$

where the Γ_k are the generators of the Clifford algebra mentioned by Kostelecky *et al.* Then we have the interesting property

$$\{\Gamma_k \otimes \sigma_+, \Gamma_l \otimes \sigma_-\} = \delta_{kl} + 2\mathrm{i}S_{kl} \otimes \sigma_3 \tag{2.36}$$

where the matrices

$$S_{kl} = -\tfrac{1}{4}\mathrm{i}[\Gamma_k, \Gamma_l] \tag{2.37}$$

generate a so(n) algebra. With respect to our constraints (2.31), the Kostelecky *et al* construction deals with a *particular* realisation given by

$$\Xi_{kl} = -2S_{kl} \otimes \sigma_3. \tag{2.38}$$

Finally let us point out that such a realisation is not unique. For example, in the $n = 2$ case, the Kostelecky *et al* realisation of our variables $\xi_{\pm,k}$ leads to 4×4 matrices while there exist 2×2 matrices which will be used in the following. In fact, this $n = 2$ case within the spin-orbit coupling procedure (already considered by Durand (1985)) will be reanalysed in § 3.3 and compared with the standard procedure in § 3.4. In the $n = 1$ case both procedures evidently coincide.

3. Symmetries and supersymmetries of the motion in a constant magnetic field

As already mentioned in the introduction, the interaction with a constant magnetic field is intimately related to the study of the harmonic oscillator. In fact, if the magnetic field is chosen along the z axis ($\boldsymbol{B} \equiv (0, 0, B)$), the interesting context corresponds to the two-dimensional harmonic oscillator described in the (x, y) plane.

Let us exploit here the results of § 2 for $n=2$ and apply them to the associated magnetic considerations which can be pointed out at the levels of symmetries and supersymmetries and also at the levels of kinematical and dynamical characteristics. We want to put in form successively the connections between the two contexts, first by limiting ourselves to symmetries (§ 3.1) and secondly by extending our considerations to supersymmetries (§§ 3.2–3.4). In fact, these last three subsections are devoted to supersymmetries (§ 3.2) implied by the standard procedure described in § 2.2, to supersymmetries (§ 3.3) implied by the spin-orbit coupling procedure described in § 2.3 and finally (§ 3.4) to the connection between the two sets of procedures and results.

3.1. Symmetries

The Schrödinger equation describing the interaction of a charged (e) particle with a *constant* magnetic field $\boldsymbol{B}=(0,0,B)$ is

$$\mathrm{i}\partial_t\Psi_{\mathrm{M}}=H_{\mathrm{M}}\Psi_{\mathrm{M}}=\tfrac{1}{2}(\underline{p}-e\underline{A})^2\Psi_{\mathrm{M}}=\tfrac{1}{2}\underline{\Pi}^2\Psi_{\mathrm{M}} \tag{3.1}$$

where the subscript M refers to the magnetic case and where underlined letters are two-dimensional vectors in the (x, y) plane. With the suitable gauge symmetric potential $\boldsymbol{A}^{\mathrm{S}}=-\frac{1}{2}\boldsymbol{r}\times\boldsymbol{B}$ leading to the *ad hoc* magnetic field, we get the Hamiltonian

$$H_{\mathrm{M}}=\tfrac{1}{2}[\underline{p}^2+\tfrac{1}{4}e^2B^2\underline{x}^2-eB(xp_y-yp_x)] \tag{3.2}$$

or by putting $eB=2\omega$ and $L\equiv L_{12}\equiv(2.7)$

$$\begin{aligned}H_{\mathrm{M}}&=\tfrac{1}{2}(\underline{p}^2+\omega^2\underline{x}^2)-\omega L\\&=H_{\mathrm{O}}-\omega L.\end{aligned} \tag{3.3}$$

The last relation clearly shows the connection between the present context and that of the harmonic oscillator characterised by (2.1).

Applying now the results of § 2.1, let us search for the corresponding results in the magnetic case. Due to the fundamental role played by the operators $P_{\pm,k}\equiv(2.4)$, it is not difficult to construct the algebra of symmetries. Indeed, if we define

$$P_{+,\pm}(t)=P_{+,1}(t)\pm\mathrm{i}P_{+,2}(t)\qquad P_{-,\pm}(t)=P_{-,1}(t)\pm\mathrm{i}P_{-,2}(t) \tag{3.4}$$

we get from the commutation relations (2.26) between H_{O}, L and the $P_{\pm,k}$, the constants of motion for H_{M}:

$$\begin{aligned}\pi_+&=\pi_x+\mathrm{i}\pi_y=\exp(\mathrm{i}\omega t)P_{+,+}(t)=P_{+,+}(0)\\ \pi_-&=\pi_x-\mathrm{i}\pi_y=\exp(-\mathrm{i}\omega t)P_{-,-}(t)=P_{-,-}(0)\\ P_+&=\exp(-2\mathrm{i}\omega t)(\Pi_x-\mathrm{i}\Pi_y)=\exp(-\mathrm{i}\omega t)P_{+,-}(t)=\exp(-2\mathrm{i}\omega t)P_{+,-}(0)\\ P_-&=\exp(2\mathrm{i}\omega t)(\Pi_x+\mathrm{i}\Pi_y)=\exp(\mathrm{i}\omega t)P_{-,+}(t)=\exp(2\mathrm{i}\omega t)P_{-,+}(0).\end{aligned} \tag{3.5}$$

They generate with the identity the Heisenberg algebra h(4) where we evidently recover the well known Johnson–Lippmann constants (Johnson and Lippmann 1949)

$$\pi_x=p_x-\omega y\qquad \pi_y=p_y+\omega x \tag{3.6}$$

and the new ones $P_\pm$ obtained by Durand (1985).

The dynamical algebra sp(4) is then directly constructed according to (2.5) as generated by all the second-order products of the h(4) operators. Inside the sp(4) algebra, let us insist on the kinematical one corresponding to so(2, 1)$\oplus$so(2) which is in fact generated by $H_O = (2.1)$, $C_\pm \equiv (2.6)$ and $L \equiv L_{12} \equiv (2.7)$. In terms of the operators (3.5), we get for the latter

$$H_O = \tfrac{1}{4}[\{\pi_+, \pi_-\} + \{P_-, P_+\}] \tag{3.7a}$$

$$C_\pm = \pm\tfrac{1}{4}\mathrm{i}\{\pi_\pm, P_\mp\} \tag{3.7b}$$

and

$$L = \frac{1}{8\omega}[\{\pi_+, \pi_-\} - \{P_+, P_-\}]. \tag{3.7c}$$

In conclusion we find (as expected) isomorphic algebras of dynamical *and* kinematical symmetries for the magnetic and harmonic oscillator cases. The kinematical [so(2, 1)⊕ so(2)] generators are unchanged but the h(4) ones do correspond to each other according to (2.4) and (3.5). A very simple way (Dehin and Hussin 1987) enlightening these results will be exploited in § 4 within a general discussion including supersymmetries.

3.2. Supersymmetries and the standard procedure

Let us now restrict § 2.2 to the $n = 2$ case and consider the supersymmetric Hamiltonian (2.10). Within a 4×4 representation of the Clifford algebra (2.13), we have in terms of the Pauli matrices (σ_i, $i = 1, 2, 3$, $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)$, $(\sigma_\pm)^\dagger = \sigma_\mp$):

$$\begin{aligned}
\varphi_1^1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_1 + \mathrm{i}\mathbb{1} \\ \sigma_1 - \mathrm{i}\mathbb{1} & 0 \end{pmatrix} & \varphi_2^1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_2 - \sigma_3 \\ \sigma_2 - \sigma_3 & 0 \end{pmatrix} \\
\varphi_1^2 &= -\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_2 + \sigma_3 \\ \sigma_2 + \sigma_3 & 0 \end{pmatrix} & \varphi_2^2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_1 - \mathrm{i}\mathbb{1} \\ \sigma_1 + \mathrm{i}\mathbb{1} & 0 \end{pmatrix}
\end{aligned} \tag{3.8}$$

or correspondingly

$$\begin{aligned}
\xi_{+,1} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_- - \frac{1}{2}\mathrm{i}(\sigma_3 - \mathbb{1}) \\ \sigma_- - \frac{1}{2}\mathrm{i}(\sigma_3 + \mathbb{1}) & 0 \end{pmatrix} & \xi_{-,1} &= (\xi_{+,1})^\dagger \\
\xi_{+,2} &= \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} 0 & \sigma_- + \frac{1}{2}\mathrm{i}(\sigma_3 - \mathbb{1}) \\ \sigma_- + \frac{1}{2}\mathrm{i}(\sigma_3 + \mathbb{1}) & 0 \end{pmatrix} & \xi_{-,2} &= (\xi_{+,2})^\dagger.
\end{aligned} \tag{3.9}$$

Such a choice will be motivated by further developments: it is essentially different from the ones introduced by others (de Crombrugghe and Rittenberg 1983, D'Hoker and Vinet 1985, Sattinger and Weaver 1986).

Here the supersymmetric Hamiltonian (2.10) can be written

$$H_O^{SS} = \tfrac{1}{2}(\underline{p}^2 + \omega^2\underline{x}^2) - \omega\begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix} = H_O + Y \tag{3.10}$$

and the total angular momentum operator (2.23) becomes

$$J = J_{12} = L + \Sigma = L + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{3.11}$$

Now, using $H_O^{SS} = (3.10)$ and $J = (3.11)$, let us construct a supersymmetric magnetic Hamiltonian by analogy with (3.3):

$$H_M^{SS} = H_O^{SS} - \omega J. \tag{3.12}$$

Such a Hamiltonian does, in fact, correspond to the amplification in the 4×4 representation of the Pauli Hamiltonian H_{P}. Indeed we have

$$H_{\mathrm{M}}^{\mathrm{SS}}=H_{\mathrm{O}}-\omega L-\omega\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix}=(H_{\mathrm{M}}-\omega\sigma_3)\otimes\mathbb{1}=H_{\mathrm{P}}\otimes\mathbb{1}. \tag{3.13}$$

Due to the structure of (3.12) and the results of § 2.2, it is straightforward to show that the MDI superalgebra corresponding to the magnetic case is again isomorphic to osp(4/4) $\oplus$ sh(4/4). The study developed in § 3.1 can immediately be extended to the supersymmetric context as it works for the harmonic oscillator (see § 2.2). The Heisenberg superalgebra sh(4/4) is generated by the identity, the four bosonic (3.5) and the four fermionic operators, the last corresponding to the $T_{\pm,k}\equiv(2.18)$ in the harmonic oscillator case, i.e.

$$\begin{aligned}
\xi_{+,+}&=\xi_{+,1}+\mathrm{i}\xi_{+,2}=\exp(\mathrm{i}\omega t)(T_{+,1}+\mathrm{i}T_{+,2})=\exp(\mathrm{i}\omega t)T_{+,+}\\
\xi_{-,-}&=\xi_{-,1}-\mathrm{i}\xi_{-,2}=\exp(-\mathrm{i}\omega t)(T_{-,1}-\mathrm{i}T_{-,2})=\exp(-\mathrm{i}\omega t)T_{-,-}\\
\eta_{+,-}&=\exp(-2\mathrm{i}\omega t)\xi_{+,-}=\exp(-2\mathrm{i}\omega t)(\xi_{+,1}+\mathrm{i}\xi_{+,2})\\
&=\exp(-\mathrm{i}\omega t)(T_{+,1}-\mathrm{i}T_{+,2})=\exp(-\mathrm{i}\omega t)T_{+,-}\\
\eta_{-,+}&=\exp(2\mathrm{i}\omega t)\xi_{-,+}=\exp(2\mathrm{i}\omega t)(\xi_{-,1}+\mathrm{i}\xi_{-,2})\\
&=\exp(\mathrm{i}\omega t)(T_{-,1}+\mathrm{i}T_{-,2})=\exp(\mathrm{i}\omega t)T_{-,+}.
\end{aligned} \tag{3.14}$$

Finally the dynamical superalgebra osp(4/4) can immediately be constructed through all the second-order products of the sh(4/4) operators as already noticed in the preceding cases.

Let us now examine more deeply the contents of this 41-dimensional dynamical superalgebra osp(4/4) $\oplus$ sh(4/4) in this magnetic context.

The *kinematical* superalgebra is [osp(2/2)$\oplus$so(2)]$\oplus$sh(4/4). If the nine sh(4/4) generators have already been explicitly mentioned (see (3.5) and (3.14)), let us recall that the osp(2/2) superalgebra does contain the operators(3.7a, b), the generator $Y\equiv(2.21)$ and the supersymmetric charges $Q_\pm\equiv(2.14)$ as well as $S_\pm\equiv(2.22)$. These last five operators can evidently be written in terms of the sh(4/4) generators (3.5) and (3.14). We get, respectively,

$$Y=\tfrac{1}{4}\omega([\xi_{+,+},\xi_{-,-}]+[\eta_{+,-},\eta_{-,+}] \tag{3.15}$$

$$Q_\pm=\frac{1}{2\sqrt{2}}(\pi_\pm\xi_{\pm,\pm}+P_\mp\eta_{\pm,\mp}) \tag{3.16}$$

and

$$S_\pm=\frac{1}{2\sqrt{2}}(P_\pm\xi_{\pm,\pm}+\pi_\pm\eta_{\pm,\mp}). \tag{3.17}$$

Moreover the so(2) generator coincides with the total angular momentum operator $J\equiv(3.11)$ which can be written

$$J=(1/8\omega)(\{\pi_+,\pi_-\}-\{P_+,P_-\})+\tfrac{1}{4}([\xi_{+,+},\xi_{-,-}]-[\eta_{+,-},\eta_{-,+}]). \tag{3.18}$$

The kinematical osp(2/2)$\oplus$so(2) generators are unchanged with respect to the harmonic oscillator case but not the sh(4/4) generators.

The dynamical superalgebra osp(4/4) is of dimension 32. It contains ten generators associated with a sp(4) algebra corresponding to the bosonic symmetries constructed through the anticommutators between the operators (3.5) (cf (2.5) in the harmonic oscillator context), six generators associated with a so(4) algebra corresponding to the fermionic symmetries constructed through the commutators between the operators (3.14) (cf (2.19) in the harmonic oscillator context) and sixteen supersymmetric generators corresponding to all the products of the operators (3.5) with (3.14).

Finally let us insist on the 'dynamical $\supset$ kinematical' inclusion (2.27) for $n=2$:

$$[\mathrm{osp}(4/4) \oplus \mathrm{sh}(4/4)] \supset [\mathrm{osp}(2/2) \oplus \mathrm{so}(2)] \oplus \mathrm{sh}(4/4).$$

Such a result will also be noticed in another way in § 4.

3.3. *Supersymmetries and the spin-orbit coupling procedure*

Let us apply the results of § 2.3 to the $n=2$ case and consider the supersymmetric Hamiltonian (2.33). Within a 2×2 representation of the algebra (2.32) we propose

$$\varphi_1^1=\varphi_2^2=\sigma_1 \qquad \varphi_2^1=-\varphi_1^2=-\sigma_2 \tag{3.19a}$$

or correspondingly, according to (2.31),

$$\xi_{+,1}=-\mathrm{i}\xi_{+,2}=\sigma_- \qquad \xi_{-,1}=\mathrm{i}\xi_{-,2}=\sigma_+ \qquad \Xi_{12}=-\Xi_{21}=\mathbb{1} \tag{3.19b}$$

leading to

$$[\xi_{+,1}, \xi_{-,1}]+[\xi_{+,2}, \xi_{-,2}]=-2\sigma_3. \tag{3.20}$$

Here the supersymmetric Hamiltonian (2.33) can be written

$$(H_{\mathrm{O}}^{\mathrm{SS}})'=\tfrac{1}{2}(\underset{\sim}{p}^2+\omega^2\underset{\sim}{x}^2)-\omega L-\omega\sigma_3=H_{\mathrm{P}}. \tag{3.21}$$

The MDI superalgebra (that we call osp(2/4)$\oplus$sh(2/4)) of the Pauli Hamiltonian (3.21) has already been determined by Durand (1985). Here let us give its contents in order to compare the two supersymmetrisation procedures. The Heisenberg superalgebra sh(2/4) is generated besides the identity by the four bosonic charges (3.5) and by *only two* fermionic charges $T_\pm$. The last ones are given by

$$T_\pm=\exp(\mp 2\mathrm{i}\omega t)\sigma_\pm. \tag{3.22}$$

They are obtained from the commutation relations between the bosonic charges (3.5) and the type-Q supercharges (2.14)

$$Q_\pm^{\mathrm{M}}=\frac{1}{\sqrt{2}}(\Pi_x\pm\mathrm{i}\Pi_y)\sigma_\mp. \tag{3.23}$$

Indeed we have

$$[Q_\pm^{\mathrm{M}}, \pi_+]=[Q_\pm^{\mathrm{M}}, \pi_-]=[Q_\pm^{\mathrm{M}}, P_\mp]=0$$

but

$$[Q_\pm^{\mathrm{M}}, P_\pm]=\pm 2\omega\sqrt{2}T_\mp.$$

Let us insist on the fact that this Heisenberg superalgebra is fundamentally different from the one obtained in § 3.2. Here the number of fermionic degrees of freedom is half of the number of bosonic ones, the explanation being subtended by the use of (2.32) in place of (2.13).

The dynamical superalgebra osp(2/4) can be constructed through all the second-order products between the sh(2/4) generators. We immediately get six new supercharges (supplemented by the two type-Q ones given by (3.23)):

$$S_{\pm}^{\mathrm{M}}=\frac{1}{\sqrt{2}}\pi_{\pm}T_{\mp}=\frac{1}{\sqrt{2}}\exp(\mp 2\mathrm{i}\omega t)(\pi_x\pm\mathrm{i}\pi_y)\sigma_{\mp} \tag{3.24}$$

$$U_{\pm}^{\mathrm{M}}=\frac{1}{\sqrt{2}}\pi_{\pm}T_{\pm}=\frac{1}{\sqrt{2}}\exp(\pm 2\mathrm{i}\omega t)(\pi_x\pm\mathrm{i}\pi_y)\sigma_{\pm} \tag{3.25}$$

and

$$V_{\pm}^{\mathrm{M}}=\frac{1}{\sqrt{2}}P_{\pm}T_{\pm}=\frac{1}{\sqrt{2}}\exp(\pm 4\mathrm{i}\omega t)(\Pi_x\pm\mathrm{i}\Pi_y)\sigma_{\pm} \tag{3.26}$$

where, for example, the type-S supercharges are associated with bosonic kinematical conformal symmetries (3.7b), etc. These six supercharges can also be obtained directly by commuting the $Q_{\pm}$ with the whole set of bosonic generators associated with the algebra sp(4). Moreover, with the fermionic charges (3.22) we also get the fermionic generator Y

$$Y=\omega[T_+,T_-]=\omega\sigma_3. \tag{3.27}$$

In conclusion, we find the 19-dimensional superalgebra osp(2/4) generated by the eight supersymmetric charges $Q_{\pm}$, $S_{\pm}$, $U_{\pm}$, $V_{\pm}$, the ten operators associated with the bosonic sp(4) algebra and the operator Y associated with the fermionic so(2) algebra.

As a final point let us mention that, in this context, the kinematical superalgebra of the Hamiltonian (3.21) is [osp(2/2)$\oplus$so(2)]$\oplus$sh(2/4) where osp(2/2) is generated by the operators (H_{O}, $C_{\pm}$) given by (3.7a, b), the supercharges (3.23) and (3.24) and the fermionic operator (3.27). The corresponding so(2) subalgebra is generated by the total angular momentum operator

$$J=L+(1/2\omega)Y=L+\tfrac{1}{2}\sigma_3 \tag{3.28}$$

while the sh(2/4) superalgebra has been previously characterised. We also notice that the supercharges $U_{\pm}$ and $V_{\pm}$ are not present in this kinematical context since through anticommutation relations they generate dynamical bosonic symmetries only as shown by Durand (1985).

3.4. Comparison between the two procedures

Through the standard procedure (§ 3.2) we have obtained the magnetic supersymmetric Hamiltonian (3.12) and have shown in connection with the harmonic oscillator context that it admits the MDI superalgebra osp(4/4)$\oplus$sh(4/4). Moreover, through the spin-orbit coupling procedure (§ 3.3) we have constructed the supersymmetric Hamiltonian (3.21) and have recovered the superalgebra osp(2/4)$\oplus$sh(2/4). These results are characteristics of 4×4 and 2×2 representations, respectively, of the corresponding algebras (2.13) and (2.32).

Due to the fact that the Hamiltonian (3.12) is nothing other than the amplification (in four dimensions) of the Pauli Hamiltonian (3.21), let us give here the connection between the two procedures as well as between the invariance superalgebras.

Starting with the supercharges $Q_\pm^M \equiv (3.23)$ ensuring (3.21), we can immediately write the Hamiltonian (3.12) as

$$\{\mathcal{Q}_+^M, \mathcal{Q}_-^M\} = H_M^{SS} \tag{3.29}$$

if we define the new 4×4 magnetic supercharges

$$\mathcal{Q}_\pm^M = \begin{pmatrix} 0 & Q_\pm^M \\ Q_\pm^M & 0 \end{pmatrix}. \tag{3.30}$$

These supercharges are related to only type-Q kinematical and dynamical supercharges of the harmonic oscillator. Indeed we get in connection with (2.20)

$$\mathcal{Q}_\pm^M = \frac{1}{\sqrt{2}}[Q_\pm \mp \mathrm{i}(Q_{\pm,12} - Q_{\pm,21})]. \tag{3.31}$$

In connection with (3.24)-(3.26), if we extend these considerations to the supercharges

$$\mathcal{S}_\pm^M = \begin{pmatrix} 0 & S_\pm^M \\ S_\pm^M & 0 \end{pmatrix} \qquad \mathcal{U}_\pm^M = \begin{pmatrix} 0 & U_\pm^M \\ U_\pm^M & 0 \end{pmatrix} \qquad \mathcal{V}_\pm^M = \begin{pmatrix} 0 & V_\pm^M \\ V_\pm^M & 0 \end{pmatrix} \tag{3.32}$$

we immediately get

$$\mathcal{S}_\pm^M = \frac{1}{\sqrt{2}}[S_\pm \mp \mathrm{i}(S_{\pm,12} - S_{\pm,21})] \tag{3.33}$$

$$\mathcal{U}_\pm^M = \frac{1}{\sqrt{2}} \exp(\pm 2\mathrm{i}\omega t)[Q_{\mp,22} - Q_{\mp,11} \pm \mathrm{i}(Q_{\mp,12} + Q_{\mp,21})] \tag{3.34}$$

and

$$\mathcal{V}_\pm^M = \frac{1}{\sqrt{2}} \exp(\pm 2\mathrm{i}\omega t)[S_{\mp,22} - S_{\mp,11} \pm \mathrm{i}(S_{\mp,12} + S_{\mp,21})]. \tag{3.35}$$

Thus, with respect to the harmonic oscillator context, the supercharges $\mathcal{Q}_\pm^M$ and $\mathcal{U}_\pm^M$ are purely of type Q while the supercharges $\mathcal{S}_\pm^M$ and $\mathcal{V}_\pm^M$ are purely of type S. In conclusion, the eight supercharges of the spin-orbit coupling procedure are doubled in the standard one and correspond to the amplified ones in the explicit form (3.30) and (3.32). These last do form a meaningful subset of the sixteen supercharges obtained in the standard procedure.

At the level of the *fundamental* superalgebras sh(4/4) and sh(2/4) associated with the standard and spin-orbit coupling procedures respectively, let us notice that the four fermionic charges (3.14) belonging to sh(4/4) are realised (with (3.9)) in the form

$$\xi_{+,+} = -\frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_3 - \mathbb{1} \\ \sigma_3 + \mathbb{1} & 0 \end{pmatrix} \qquad \xi_{-,-} = \frac{\mathrm{i}}{\sqrt{2}} \begin{pmatrix} 0 & \sigma_3 + \mathbb{1} \\ \sigma_3 - \mathbb{1} & 0 \end{pmatrix} \tag{3.36}$$

and

$$\eta_{+,-} = \sqrt{2} \exp(-2\mathrm{i}\omega t) \begin{pmatrix} 0 & \sigma_- \\ \sigma_- & 0 \end{pmatrix} \qquad \eta_{-,+} = \sqrt{2} \exp(2\mathrm{i}\omega t) \begin{pmatrix} 0 & \sigma_+ \\ \sigma_+ & 0 \end{pmatrix}. \tag{3.37}$$

Thus only the last ones (3.37) can immediately be written in terms of the two fermionic charges $T_\pm$ given by (3.22) associated with the algebra sh(2/4). In fact we have

$$\eta_{\pm,\mp} = \sqrt{2} \begin{pmatrix} 0 & T_\mp \\ T_\mp & 0 \end{pmatrix} \tag{3.38}$$

explaining the inclusion

$$\mathrm{sh}(4/4) \supset \mathrm{sh}(2/4). \tag{3.39}$$

Through these elements, it is then clear that the eight supercharges (3.30) and (3.32) do effectively refer to only the fermionic charges (3.38). Indeed we have

$$\begin{aligned} \mathcal{Q}_{\pm}^{\mathrm{M}} &= \tfrac{1}{2} P_{\mp} \eta_{\pm,\mp} & \mathcal{S}_{\pm}^{\mathrm{M}} &= \tfrac{1}{2} \pi_{\pm} \eta_{\pm,\mp} \\ \mathcal{U}_{\pm}^{\mathrm{M}} &= \tfrac{1}{2} \pi_{\pm} \eta_{\mp,\pm} & \mathcal{V}_{\pm}^{\mathrm{M}} &= \tfrac{1}{2} P_{\mp} \eta_{\mp,\pm}. \end{aligned} \tag{3.40}$$

Moreover only one charge (generating an so(2) algebra) of the so(4) dynamical algebra of fermionic symmetries (cf (2.19)) can be expressed in terms of (3.38). In conclusion, we then have explained the inclusion

$$[\mathrm{osp}(4/4) \oplus \mathrm{sh}(4/4)] \supset [\mathrm{osp}(2/4) \oplus \mathrm{sh}(2/4)]. \tag{3.41}$$

4. A simple change of variables

At the level of symmetries, we have already proposed (Dehin and Hussin 1987) a very simple change of variables enlightening the connection between the two-dimensional harmonic oscillator and the motion in a constant magnetic field (chosen along the z axis as in § 3). It corresponds to a time-dependent rotation R in the (x, y) plane defined by

$$\begin{pmatrix} x^0 \\ y^0 \end{pmatrix} = R(\omega t, e_3) \begin{pmatrix} x^{\mathrm{M}} \\ y^{\mathrm{M}} \end{pmatrix} = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} x^{\mathrm{M}} \\ y^{\mathrm{M}} \end{pmatrix} \tag{4.1}$$

when $2\omega = eB$.

Such a very clear point of view can evidently work also in the supersymmetric context. It can help us to underline correspondences in dynamical and kinematical symmetries in supersymmetric quantum mechanics. Let us immediately point out that the previously mentioned connection is in the supersymmetric context obtained if we ask for an action of our rotation on bosonic as well as on fermionic variables. This demand can immediately be understood through the superspace formulation (Bouquiaux *et al* 1987) of such problems where the bosonic and fermionic variables have to be combined in superfields of the type

$$Z_k(t) = x_k(t) + \mathrm{i}\theta \bar{\Psi}_k(t) + \mathrm{i}\bar{\theta}\Psi_k(t) + \theta\bar{\theta}F_k(t) \qquad k = 1, 2$$

if a $N = 2$ supersymmetric theory has to be constructed.

Let us first consider the *bosonic* symmetries which are fundamentally subtended by the four operators $P_{\pm,k}$ $(k = 1, 2)$ given by (2.4) in the harmonic oscillator context and by the four operators $\pi_{\pm}$ and $P_{\pm}$ given by (3.5) in the magnetic case. It is not difficult to show that they correspond to each other through the rotation (4.1). We effectively have

$$P_{\pm,1}^0 \pm \mathrm{i} P_{\pm,2}^0 \leftrightarrow \pi_x \pm \mathrm{i}\pi_y = \pi_{\pm} \tag{4.2}$$

and

$$P_{\pm,1}^0 \mp \mathrm{i} P_{\pm,2}^0 \leftrightarrow \exp(\mp 2\mathrm{i}\omega t)(\Pi_x \mp \mathrm{i}\Pi_y) = P_{\pm}. \tag{4.3}$$

Consequently all the dynamical sp(4) symmetries do correspond to each other in both cases since they are products of the h(4) generators. Moreover we also get here a better understanding of the fact that only those which are associated with so(2, 1)$\oplus$so(2) (the kinematical ones) are unchanged because they are invariant under the rotation $R(\omega t, e_3)$.

Let us then consider the *fermionic* symmetries which are fundamentally subtended by the operators $T_{\pm,k}$ given by (2.18) in the harmonic oscillator context and by the operators $\xi_{\pm,\pm}$ and $\eta_{\pm,\mp}$ given by (3.14) in the magnetic case. The correspondence is here realised by rotating the fermionic variables $\xi_{\pm,k}$. Indeed we have

$$\begin{pmatrix} \xi_{\pm,1} \\ \xi_{\pm,2} \end{pmatrix} \leftrightarrow R(\omega t, e_3) \begin{pmatrix} \xi_{\pm,1} \\ \xi_{\pm,2} \end{pmatrix} \tag{4.4}$$

so that

$$\xi_{\pm,1} \pm \mathrm{i}\xi_{\pm,2} \leftrightarrow \exp(\pm \mathrm{i}\omega t)(\xi_{\pm,1} \pm \mathrm{i}\xi_{\pm,2})$$

and

$$\xi_{\pm,1} \mp \mathrm{i}\xi_{\pm,2} \leftrightarrow \exp(\mp \mathrm{i}\omega t)(\xi_{\pm,1} \mp \mathrm{i}\xi_{\pm,2}).$$

Hence we get

$$T_{\pm,1} \pm \mathrm{i}\, T_{\pm,2} \leftrightarrow \xi_{\pm,\pm} \tag{4.5}$$

and

$$T_{\pm,1} \mp \mathrm{i}\, T_{\pm,2} \leftrightarrow \eta_{\pm,\mp}. \tag{4.6}$$

The dynamical so(4) symmetries do correspond to each other in both cases and we also understand why only the kinematical operator $Y \equiv (2.22)$ is unchanged.

Finally, through our understanding of the supercharges (with the help of the results (2.20)) as well as through the correspondences (4.2), (4.3), (4.5) and (4.6) it is evident that all the supercharges do correspond to each other in both the harmonic oscillator and magnetic contexts. Since the *same* rotation acts on the bosonic and fermionic generators, the 'kinematical' supercharges are unchanged (cf (2.30)).

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References

Balantekin A B 1985 *Ann. Phys., NY* **164** 277-87
Barut A O 1964 *Phys. Rev.* **135** B839-42
—— 1965 *Phys. Rev.* **139** B1107-12
—— 1972 *Dynamical Groups and Generalised Symmetries in Quantum Theory* (Christchurch, New Zealand: University of Canterbury Publications)
Beckers J and Hussin V 1986 *Phys. Lett.* **118A** 319-21
Beckers J, Dehin D and Hussin V 1987 *J. Phys. A: Math. Gen.* **20** 1137-54
Bouquiaux L, Dauby P and Hussin V 1987 *J. Math. Phys.* **28** 477-84
Boyer C P 1974 *Helv. Phys. Acta* **47** 590-605
Burdet G and Perrin M 1975 *J. Math. Phys.* **16** 1692-703
Burdet G, Perrin M and Sorba P 1973 *Commun. Math. Phys.* **34** 85-90
de Crombrugghe M and Rittenberg V 1983 *Ann. Phys., NY* **151** 99-126

Dehin D and Hussin V 1987 *Helv. Phys. Acta* **60** 552-9
D'Hoker E and Vinet L 1985 *Nucl. Phys.* B **260** 79-102
D'Hoker E, Vinet L and Kostelecky V A 1987 *Dynamical Groups and Spectrum Generating Algebras* ed A Barut, A Bohm and Y Ne'eman (Singapore: World Scientific)
Domokos G 1972 *Commun. Math. Phys.* **26** 15-23
Durand S 1985 *Supersymétries des systèmes mécaniques non relativistes en une et deux dimensions. Maîtrise es Sciences* Université de Montréal
Fubini S and Rabinovici E 1984 *Nucl. Phys.* B **245** 17-44
Gamboa J and Zanelli J 1985 *Phys. Lett.* **165B** 91-3
Gilmore R 1974 *Lie Groups, Lie Algebras and Some of Their Applications* (New York: Wiley)
Hagen C R 1972 *Phys. Rev.* D **5** 377-88
Jacobson N 1962 *Lie Algebras* (New York: Dover)
Johnson M H and Lippmann B A 1949 *Phys. Rev.* **76** 828-32
Kastrup H A 1962 *Ann. Phys., Lpz.* **9** 388-428
Kostelecky V A, Nieto M M and Truax D R 1985 *Phys. Rev.* D **32** 2627-33
Mack G and Salam M 1969 *Ann. Phys., NY* **53** 174-202
Miller W 1968 *Lie Theory and Special Functions* (New York: Academic)
Niederer U 1972 *Helv. Phys. Acta* **45** 802-10
—— 1973 *Helv. Phys. Acta* **46** 191-200
—— 1974 *Helv. Phys. Acta* **47** 167-72
Sattinger D H and Weaver O L 1986 *Lie Groups and Algebras with Applications to Physics, Geometry and Mechanics* (Berlin: Springer)
Talman J D 1968 *Special Functions, a Group Theoretical Approach* (New York: Benjamin)
Wigner E P 1939 *Ann. Math.* **40** 149-204
Witten E 1981 *Nucl. Phys.* B **188** 513-54
Wybourne B 1974 *Classical Groups for Physicists* (New York: Wiley)